

NONLINEAR DYNAMIC STABILITY OF SPACE VEHICLES ENTERING OR LEAVING AN ATMOSPHERE

E. V. LAITONE

Professor of Aeronautical Sciences

and

T. J. COAKLEY

Graduate Research Engineer

University of California, Berkeley

ABSTRACT

Theoretical analyses are made for the linearized phugoid oscillations of powered vehicles in steady level flight and for the nonlinear short period oscillations of unpowered vehicles on ascending or descending flight paths. It is found that the atmospheric density gradient will produce a large decrease in the classical phugoid period and that this effect increases with velocity until orbital speeds are approached.

The linearized short period oscillations are analyzed using both body axes and axes tangent to the flight path; the resulting linearized equations with time-dependent coefficients are shown to coincide only if the effect of acceleration is properly considered in the replacement of $C_{m\dot{w}}$ by $C_{m\alpha}$. The nonlinear oscillations are governed by a similar equation, except that now the coefficients are functions of the angle of attack and its derivative. A new expansion procedure is introduced which enables the approximate analysis of this nonlinear time-dependent equation to be made. It is found that the instantaneous frequency ω and the dynamic stability factor \dot{A}/A , which are constant in the usual case of constant coefficient aerodynamics, become functions both of time and the oscillation amplitude A .

SYMBOLS

- (A, θ) = oscillation amplitude and argument $\int \omega ds$ in angle of attack oscillation
 B = moment of inertia about y axis

- C_D, C_L, C_m, C_c, C_N = dimensionless aerodynamic coefficients; see Table I
 (C_i, D_{jk}) = dimensionless coefficients in expansion of nonlinear static and damping characteristics in α and α'
 $F = U/\sqrt{gR}$ = ratio of flight speed to circular orbit velocity
 g = acceleration of gravity, ft/sec²
 L, S = reference length and area
 m = vehicle mass, lb/(ft/sec²)
 q = pitching velocity
 $r = R_0 + y$ = distance from earth's center to vehicle, ft
 R = altitude of vehicle above earth's surface, ft
 y = distance from earth's surface to steady state orbit of vehicle, ft; see Fig. 1
 s = dimensionless distance traveled along trajectory; $s = \int V dt/L$
 V = flight velocity, ft/sec
 (u, w) = axial and normal velocity components of velocity in body axes, ft/sec
 X, Z = axes fixed in body
 X', Z' = axes fixed in space with origin at center of earth
 \bar{u} = horizontal component of flight velocity, ft/sec
 $\alpha, \gamma, \theta, \varphi$ = (See Fig. 1)
 ρ = air density, (mass/ft³)
 $\sigma = mL^2/B, \mu = \rho SL/2m, \beta = \rho'/\rho$

EQUATIONS OF MOTION

Using the axes system that is always tangent to the actual flight path, Allen (1957) presented the linearized equations of motion for a hypersonic nonrolling missile having a longitudinal plane of symmetry and trimmed to follow a nearly zero-lift trajectory. The equations used by Allen to predict the short-period longitudinal oscillations during high rates of acceleration or deceleration may be written as follows (e.g., see Sommer and Tobak, 1959):

$$\begin{aligned}
 m\dot{V} &= -\frac{1}{2}\rho V^2 SC_D - mg \sin \gamma + T \cos \alpha \\
 mV\dot{\gamma} &= \frac{1}{2}\rho V^2 SC_L - m \left(g - \frac{V^2}{r} \right) \cos \gamma + T \sin \alpha \\
 B(\ddot{\alpha} + \ddot{\gamma} - \ddot{\varphi}) &= \frac{1}{2}\rho V^2 SLC_m \\
 \dot{r} &= V \sin \gamma; \quad \dot{\varphi} = \frac{V \cos \gamma}{r}
 \end{aligned} \tag{1}$$

In these equations C_D, C_L, C_m are the nondimensional drag, lift, and moment coefficients (see Table I) and are, in general, nonlinear functions of α, α' , and θ . The other symbols and variables are illustrated in Fig. 1.

The second set of equations with which we will be concerned is written in terms of axes fixed in the vehicle so as to coincide with the principal body axes, as (e.g., see Laitone, 1959):

$$\begin{aligned}
 m(\dot{u} + wq) &= T - \frac{1}{2} \rho V^2 SC_c - mg \sin \theta \\
 m(\dot{w} - uq) &= -\frac{1}{2} \rho V^2 SC_N + mg \cos \theta \\
 B\dot{q} &= \frac{1}{2} \rho V^2 SLC_m \\
 q &= \dot{\theta} - \dot{\varphi} \approx \dot{\theta} - \frac{u}{r} \\
 V &= \sqrt{u^2 + w^2}; \quad w = V \sin \alpha = u \tan \alpha
 \end{aligned}
 \tag{2}$$

The variables and symbols in these equations are illustrated in Fig. 1, and the derivatives of the aerodynamic coefficients are presented in Table I.

For the tangent axes system of Eq. (1), Allen (1957) gave the linearized equation of motion for a hypersonic nonrolling reentry missile having a longitudinal plane of symmetry and trimmed to follow a nearly zero-lift trajectory. Allen also assumed constant aerodynamic stability derivatives, negligible gravitational acceleration, and an exponential density variation with altitude.

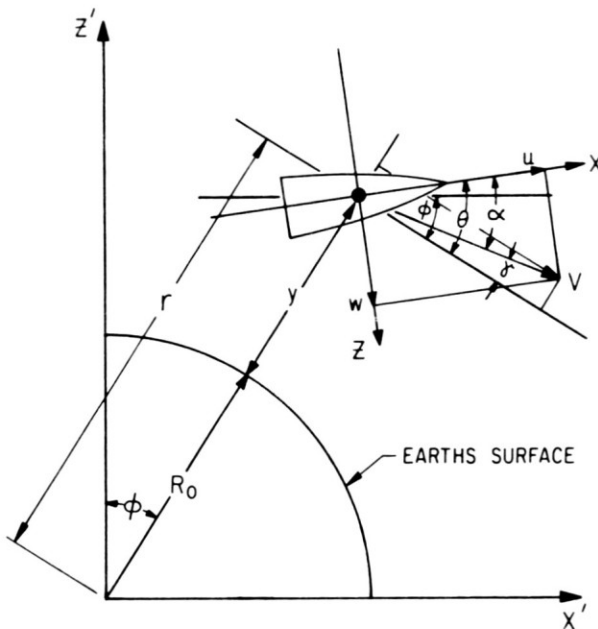


Fig. 1.

This equation which governs the linearized short period oscillations in angle of attack of a reentry missile can be written in dimensionless form as follows:

$$\begin{aligned}\alpha''(s) + \mu D_1 \alpha'(s) + \mu C_1 \alpha(s) &= 0 \\ D_1 &= [C_{L\alpha} - C_D - \sigma (C_{mq} + C_{m\dot{\alpha}})] \\ C_1 &= \left[-\sigma (C_{m\alpha} + \mu C_{mq} C_{L\alpha}) + C_{L\alpha} \frac{\rho'}{\rho} - \mu C_{L\dot{\alpha}} C_D \right]\end{aligned}\quad (3)$$

where the independent variable s represents the number of body lengths traveled along the trajectory and is related to time by the following relations:

$$\begin{aligned}S &= \frac{1}{L} \int_{\sigma}^t V(\tau) d\tau; \quad \mu(s) = \frac{\rho(s)SL}{2m} \ll 1 \\ \frac{d(\quad)}{dt} &= \frac{V}{L} \frac{d(\quad)}{ds}; \quad \frac{d^2(\quad)}{dt^2} = \left(\frac{V}{L}\right)^2 \frac{d^2(\quad)}{ds^2} + \frac{V}{L^2} \frac{dV}{ds} \frac{d(\quad)}{ds}\end{aligned}\quad (4)$$

The same analysis was done by Laitone (1959) using the body axes system of Eq. (2). By making essentially the same assumptions as Allen, Laitone obtained an equation in terms of the normal velocity w , rather than the angle of attack in the following form:

$$\begin{aligned}w''(s) + \mu D_2 w'(s) + \mu C_2 w(s) &= 0 \\ D_2 &= \left[V \frac{\partial C_N}{\partial w} - \sigma \frac{V^2}{L} \frac{\partial C_m}{\partial \dot{w}} - \sigma \frac{V}{L} \frac{\partial C_m}{\partial q} \right] \\ C_2 &= \left[-\sigma V \left(\frac{\partial C_m}{\partial w} + \mu \frac{V}{L} \frac{\partial C_m}{\partial q} \frac{\partial C_N}{\partial w} \right) + \frac{1}{\rho} \frac{d(\rho V C_{Nw})}{ds} \right]\end{aligned}\quad (5)$$

This equation can now be shown to be identical with that of Allen if the correct values of $C_{m\alpha}$ and $C_{m\dot{\alpha}}$ are introduced in place of $\partial C_m/\partial w$ and $\partial C_m/\partial \dot{w}$. This is accomplished as follows:

$$w = \alpha V; \quad w' = \alpha' V + \alpha V'; \quad w'' = \alpha'' V + 2\alpha' V' + \alpha V'' \quad (6)$$

and (noting that $C_{m\alpha}$, etc. are dimensionless, whereas $\partial C_m/\partial w$, etc. are not)

$$\begin{aligned}dC_m(w, \dot{w}) &= \left[\frac{\partial C_m}{\partial w} dw + \frac{\partial C_m}{\partial \dot{w}} d\dot{w} \right]_{w=0=\dot{w}} = \left[\frac{\partial C_m}{\partial w} (V d\alpha + \alpha dV) \right. \\ &+ \left. \frac{\partial C_m}{\partial w} (\dot{\alpha} dV + V d\dot{\alpha} + \alpha d\dot{V} + \dot{V} d\alpha) \right]_{\alpha=0=\dot{\alpha}} = \left[\left(V \frac{\partial C_m}{\partial w} + \dot{V} \frac{\partial C_m}{\partial w} \right) d\alpha \right. \\ &\quad \left. + \left(V \frac{\partial C_m}{\partial w} \right) d\dot{\alpha} \right]_{t=0=\alpha=\dot{\alpha}}\end{aligned}$$

so that

$$C_{m\alpha} = \frac{\partial C_m}{\partial \alpha} = V \frac{\partial C_m}{\partial w} + \dot{V} \frac{\partial C_m}{\partial w}$$

$$C_{m\alpha} = \frac{V}{L} \frac{\partial C_m}{\partial \alpha} = \frac{V^2}{L} \frac{\partial C_m}{\partial w}; \quad V \frac{\partial C_m}{\partial w} = C_{m\alpha} - \frac{\dot{V}L}{V^2} C_{m\alpha} \quad (7)$$

The last set of equations shows that $C_{m\alpha}$ is no longer given directly by C_{m_w} in accelerated motion. This is a very important point since the theoretical calculations actually yield C_{m_w} and C_{m_w} directly, and in the past these have been assumed to give $C_{m\alpha}$ and $C_{m\alpha}$ precisely.

By inverting Eq. (7) for C_{m_w} and C_{m_w} in terms of $C_{m\alpha}$ and $C_{m\alpha}$, employing the velocity equation which relates V to C_D under the assumption of negligible gravity (i.e., $V'/V = -\mu C_D$), and expressing C_N in terms of C_L and C_D (i.e., $C_{N\alpha} = C_{L\alpha} + C_D$), it is easily shown that Laitone's equation, i.e., Eq. (5), reduces to Eq. (3) or Allen's result.

It is also of interest to inspect the explicit expression for the variation of the pitching velocity q during the short period longitudinal oscillations, which can be obtained from Eq. (5), if we assume that C_{m_w} is negligible so that

$$q''(s) + \mu D_3 q'(s) + \mu C_3 q(s) = 0$$

$$D_3 = [C_{N\alpha} - \sigma C_{mq}] - \frac{\rho'}{\rho\mu}$$

$$C_3 = \sigma[-C_{m\alpha} - \mu C_{mq} C_{N\alpha}] \quad (8)$$

Fortunately, C_{m_w} can be very small in hypersonic flow; consequently, this expression for the pitching velocity is valid for this important case. A comparison of Eqs. (9) and (8) shows that a high rate of deceleration (or large C_D) can make the angle of attack α oscillations become divergent or even unstable, while the pitching velocity q by itself remains stable. Therefore, the remainder of the analysis will be concerned only with the oscillations in the angle of attack α .

For vehicles which are describing arbitrary paths through the atmosphere, Allen's equations are not strictly applicable because of the assumption of a nearly straight trajectory. For this more general case, Sommer and Tobak (1959) have derived the linearized equations describing the short period oscillations about the mean trajectory motion. These authors start with the general equations of motion in tangent axes, i.e., Eq. (1), and under the assumption that they have been solved for the static or steady state trajectory, take small perturbations in the angular variables about their mean or static values. By separating these equations into perturbed and unperturbed components, subtracting the unperturbed set out since they are identically satisfied, and then

nondimensionalizing, they are able to reduce the equations describing the short-period oscillations in angle of attack to the following equation:

$$\alpha''(s) + \mu D_4 \alpha'(s) + \mu C_4 \alpha(s) = 0$$

$$D_4 = [C_{L\alpha} - \sigma(C_{mq} + C_{m\dot{\alpha}})] + \frac{V'(s)}{\mu V}$$

$$C_4 = \left[-\sigma(C_{m\alpha} + \mu C_{mq} C_{L\alpha}) + \left(\frac{\mu'}{\mu} + \frac{V'}{V} \right) C_{L\alpha} \right] \quad (9)$$

Since μ is a very small quantity, all of the terms in C_4 may be neglected by comparison with the static moment term [see, e.g., Allen (1957)], and the following basic equation is obtained which is analyzed extensively by Sommer and Tobak (1959):

$$\alpha''(s) + \left\{ \frac{V'}{V} + \mu [C_{L\alpha} - \sigma(C_{mq} + C_{m\dot{\alpha}})] \right\} \alpha' - \mu \sigma C_{m\dot{\alpha}} \alpha = 0 \quad (10)$$

Now for nonlinear aerodynamics, that is aerodynamic coefficients which are functions of α , α' , and θ , it can be shown (although the derivation is quite lengthy and, therefore, will be omitted here) that under the assumptions of Sommer and Tobak a nonlinear equation can be obtained for the short period angle of attack oscillations having a form nearly identical with Eq. (10). But with C_{mq} , $C_{m\dot{\alpha}}$ now dependent on α and α' , and C_L and $C_{m\alpha}$ dependent on α , these coefficients can also be functions of s .

By letting

$$D(\alpha, \alpha', s) = \frac{V'}{\mu V} + \frac{dC_L}{d\alpha} - \sigma(C_{mq} + C_{m\dot{\alpha}}) \quad (11)$$

$$C(\alpha, s) = -\sigma C_{m\alpha}(\alpha, s) \quad (12)$$

Eq. (10) can be written in the more compact form

$$\alpha''(s) + \mu D(\alpha, \alpha', s) \alpha' + \mu C(\alpha, s) \alpha = 0 \quad (13)$$

We observe that for vehicles traveling at high speeds wherein the drag deceleration term ($-\mu C_D$) is relatively large compared to the gravitational deceleration term ($gL \sin \gamma / V^2$), Eq. (11) for $D(\alpha, \alpha', s)$ can be simplified to the form of D_1 in Eq. (3). At low altitudes where the vehicle has attained a nearly steady-state glide velocity, or for a vehicle on a skip trajectory, the relation $V'/V = -\mu C_D$ is not valid, and Eq. (11) must be used for $D(\alpha, \alpha', s)$.

A useful form of Eq. (13) can be derived for high-speed vehicles ($V'/V \approx -\mu C_D$) descending or ascending through an exponential atmosphere ($\mu = \mu_0 e^{-\beta y}$) and over path segments short enough to consider the flight-path angle γ constant. The dimensionless form of this equation is easily obtained from Eq. (13) and is

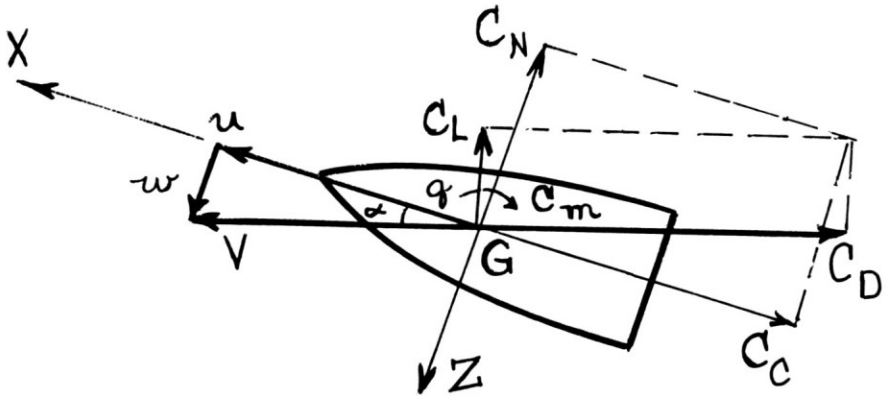
$$\alpha''(r) + \epsilon e^{-\epsilon Kr} D(\alpha, \alpha', \epsilon r) \alpha'(r) + e^{-\epsilon Kr} C(\alpha, \epsilon r) \alpha(r) = 0 \quad (14)$$

where ϵ , K , and r are defined as follows:

$$\epsilon = \sqrt{\mu_m} = \sqrt{\mu_0 e^{-1/2\beta y_0}}, \quad K = \left(\frac{\beta L \sin \gamma}{\mu_m} \right), \quad r = \epsilon S = \sqrt{\mu_m} S \quad (15)$$

and where y_0 is a nominal altitude. At all altitudes ϵ , being a measure of the ratio of air density to mean body density (since $\mu = \beta SL/2m = \rho_a/2\rho_b$), is almost always a small parameter of $O(10^{-1} - 10^{-2})$. In addition, for altitudes y_0 below a certain maximum, or for small enough flight-path angles γ , the parameter ϵK is also small, i.e., $\epsilon K \ll 1$.

Table I. AERODYNAMIC STABILITY DERIVATIVES FOR THE LONGITUDINAL MOTION OF A MISSILE WITH CONSTANT THRUST



$$C_L = C_N \cos \alpha - C_C \sin \alpha$$

$$C_D = C_c \cos \alpha + C_N \sin \alpha$$

$$C_m = C_{m_{nose}} - \frac{G}{L} C_N \quad M(G) = \frac{1}{2} \rho V^2 S L C_m$$

$$-X = [\frac{1}{2} \rho V^2 S (C_D \cos \alpha - C_L \sin \alpha)] = \frac{1}{2} \rho V^2 S C_c$$

$$-Z = [\frac{1}{2} \rho V^2 S (C_L \cos \alpha + C_D \sin \alpha)] = \frac{1}{2} \rho V^2 S C_N$$

$$X_u = \left(\frac{\partial X}{\partial u} \right)_{\alpha=0=w} = -\rho u S \left[C_C + \frac{u}{2} \frac{\partial C_C}{\partial u} + \frac{u C_C}{2\rho} \frac{\partial \rho}{\partial u} \right]_{\alpha=0=w}$$

$$X_w = \left(\frac{\partial X}{\partial w} \right)_{\alpha=0=w} = -\rho u S \left[\frac{u}{2} \frac{\partial C_C}{\partial w} \right]_{\alpha=0=w} = -\rho \frac{u S}{2} \left[\frac{\partial C_C}{\partial \alpha} \right]_{\alpha=0=i}$$

$$= -\rho \frac{u S}{2} \left[\frac{\partial C_D}{\partial \alpha} - C_L \right]_{\alpha=0=i}$$

$$X_q = \left(\frac{\partial X}{\partial q} \right)_{q=0} = -\rho u S \left[\frac{L}{2} \left(\frac{\partial C_c}{u \partial q} \right) \right]_{q=0}$$

$$\rho(r) = \rho(R_0) \exp [-\beta(r - R_0)]; \quad \frac{\partial \rho}{\partial r} = -\beta \rho; \quad \frac{\partial p}{\partial r} = -g \rho$$

$$\left(\frac{\partial C_c}{\partial u} \right)_{\alpha=0} = \left(\frac{\partial C_D}{\partial u} \right)_{\alpha=0} = \left[\frac{1}{a} \frac{\partial C_D}{\partial M} \left(1 - M \frac{\partial a}{\partial r} / \frac{du}{dr} \right) \right]_{\alpha=0}$$

$$M = \left(\frac{V}{a} \right)_{\alpha=0} = \left(\frac{u}{a} \right)_{\alpha=0}; \quad a = \sqrt{\frac{\gamma p}{\rho}}; \quad \frac{\partial \rho}{\partial u} = \left(\frac{\partial \rho}{\partial r} \right) \frac{dr}{du} = \frac{-\beta \rho}{\left(\frac{du}{dr} \right)}$$

$$\frac{\partial a}{\partial r} = \frac{a}{2} \left(-\frac{1}{\rho} \frac{\partial \rho}{\partial r} + \frac{1}{p} \frac{\partial p}{\partial r} \right) = \frac{a}{2} \left(\beta - \frac{\rho g}{p} \right)$$

$$Z_u = \left(\frac{\partial Z}{\partial u} \right)_{\alpha=0=w} = -\rho u S \left[C_N + \frac{u}{2} \frac{\partial C_N}{\partial u} + \frac{u C_N}{2\rho} \frac{\partial \rho}{\partial u} \right]_{\alpha=0=w}$$

$$Z_w = -\rho u S \left[\frac{u}{2} \frac{\partial C_N}{\partial w} \right]_{\alpha=0=w} = -\frac{\rho u S}{2} \left[\frac{\partial C_L}{\partial \alpha} + C_D \right]_{\alpha=0=i}$$

$$Z_q = -\rho u S \left[\frac{L}{2} \left(\frac{\partial C_N}{u \partial q} \right) \right]_{q=0} = -\rho u S \left[\frac{L}{2} \left(\frac{\partial C_L}{u \partial q} \right) \right]_{q=0=\alpha}$$

$$\left(\frac{\partial C_N}{\partial u} \right)_{\alpha=0} = \left(\frac{\partial C_L}{\partial u} \right)_{\alpha=0} = \left[\frac{1}{a} \frac{\partial C_L}{\partial M} \left(1 - M \frac{\partial a}{\partial r} / \frac{du}{dr} \right) \right]_{\alpha=0}$$

$$M_u = \left(\frac{\partial M}{\partial u} \right)_{\alpha=0=w} = \rho u S L \left[C_m + \frac{u}{2} \frac{\partial C_m}{\partial u} + \frac{u C_m}{2\rho} \frac{\partial \rho}{\partial u} \right]_{\alpha=0}$$

$$M_w = \rho u S L \left[\frac{u}{2} \frac{\partial C_m}{\partial w} \right]_{\alpha=0=w} = \rho u S L \left[\frac{1}{2} \frac{\partial C_m}{\partial \alpha} \right]_{\alpha=0=i}$$

$$M_q = \rho u S L \left[\frac{L}{2} \left(\frac{\partial C_m}{u \partial q} \right) \right]_{q=0} = \frac{\rho u S L^2}{2} C_{m_q}$$

$$M_w = \rho u S L \left[\frac{u}{2} \frac{\partial C_m}{\partial w} \right]_{w=0} = \rho S L^2 \left[\frac{\partial C_m}{u \partial \dot{\alpha}} \right]_{\alpha=0=\dot{\alpha}}$$

$$C_{m\alpha} = \left(\frac{\partial C_m}{\partial \alpha} \right)_{\alpha=0} = u \frac{\partial C_m}{\partial w} + \dot{u} \frac{\partial C_m}{\partial \dot{w}}$$

$$C_{m\dot{\alpha}} = \left(\frac{\partial C_m}{\partial \dot{\alpha}} \right)_{\alpha=0=\dot{\alpha}} = \frac{u^2}{L} \frac{\partial C_m}{\partial \dot{w}}$$

SHORT-PERIOD OSCILLATIONS OF ASCENDING OR DESCENDING VEHICLES

In this section we will investigate the combined effects of nonlinear aerodynamic forces and moments and varying air density on the short period oscillations of vehicles ascending or descending through the atmosphere. The basic equations we will study are Eqs. (13) and (14) and in the appendix we have developed a general method for treating equations of this type. This method is similar to the method of Krylov and Bogoliubov (1943) which is applicable to autonomous systems having small nonlinear perturbations (i.e., quasi-linear systems) and to the extension of this method by Bogoliubov and Mitropolsky (1955) to nonautonomous systems with slowly varying time-dependent parameters. However, our technique differs somewhat from theirs in that it can be used to analyze systems with comparatively large conservative nonlinearities [i.e., in static moment $C(\alpha, s)$ α]. Murphy (1957) and (1962) has employed a similar technique, which he calls the "quasi-linear substitution" method, to investigate the nonlinear time-dependent oscillations of spinning symmetric missiles, and in the simplification to longitudinal or planar motion his results and ours for the amplitude and frequency characteristics essentially agree. Murphy's results are restricted, however, to the discussion of symmetric missiles and in particular do not describe the effects of nonlinear asymmetries in the static moment on the oscillatory motion. For this reason and others, we have developed the present theory which accounts for these effects and provides an algorithm for calculating higher approximations to the motion.

The content of this section will be divided into four basic parts. In the first part general formulas will be developed which give the frequency and amplitude characteristics and the higher harmonics in the response of the motion in angle of attack. In the second part we will briefly compare our results for the linearized theory with those of other authors, namely Allen (1957) and Sommer and Tobak (1959). In the third part we will investigate the effects of varying air density on vehicles with nonlinear damping moments (Van der Pol type) and will determine specifically the effect that this produces on the classic limit cycle of Van der Pol which exists for time-independent parameters. In the final part we will investigate the effects of nonlinear static moments on the oscillation and specifically will produce results for quadratic and cubic nonlinearities in $C(\alpha, s)$ α .

Before proceeding to the general equations of the first approximation, we must add a note of caution in the results of our analysis. This is that the oscillations are assumed to be longitudinal, taking place only in a plane. As a matter of fact, this assumption is not always valid on account of the possible nonlinear

coupling between longitudinal and lateral modes which could cause the longitudinal motion to be unstable with respect to small perturbations in the lateral mode. This nonlinear coupling has been beautifully illustrated, experimentally and numerically, by Waldron and Cheers in these very proceedings and is one of the most important theoretically unsolved problems in the dynamics of missiles and space vehicles. It is hoped that the methods developed in the appendix to this work can be logically and practically extended to cope with this important problem.

FIRST APPROXIMATION TO THE SOLUTION OF THE GENERAL EQUATION OF SHORT PERIOD OSCILLATIONS

As described in the appendix, it is possible to introduce a transformation of variables $(\alpha, \alpha') \rightarrow (A, \theta)$ such that the original differential equation for α , i.e., Eq. (13), is transformed into a system for A' and θ' . By taking the transformation in the form

$$\alpha = A \cos \theta + U(A, \theta, s), \quad \alpha' = -\omega [A \sin \theta + V(A, \theta, s)]$$

assuming $A' = \delta(A, s)A$ and $\theta' = \omega(A, s)$ to be dependent only upon A and s , and that U/A , V/A , and δ/ω are small with respect to one, the first approximation to this transformed system is as follows:

$$\alpha \approx A \left\{ \cos \theta - \frac{1}{C_1^*(A, s)} \sum_{n=0}^{\infty} \frac{1}{1-n^2} C_n^*(A, s) \cos n\theta \right\} + O(\sqrt{\mu})$$

$$\theta' = \omega \approx \sqrt{\mu C_1^*(A, s)} \frac{A'}{A} = \delta \approx - \left(\frac{4C_1^*}{4C_1^* + C_{1A}^* A} \right) \left(\frac{(\mu C_1^*)_s}{4\mu C_1^*} + \frac{1}{2} \mu D_1^*(A, s) \right) \quad (16)$$

In the α equation, the Σ^* is meant to exclude $n = 1$ from the sum. A similar but more complex relation exists for α' but will not be presented here. In these equations terms of order $\sqrt{\mu}$ have been omitted from α , and terms of order μ^2 and higher have been omitted from ω^2 and δ . The parameters C_n^* and D_n^* are given by the Fourier expansions of $C(\alpha, s)\alpha$ and $D(\alpha, \alpha', s)\alpha'$ and are as follows, for $n \neq 0$:

$$C_n^*(A, s) = \frac{1}{\pi} \int_0^{2\pi} C(A \cos \theta, s) \cos \theta \cos n\theta \, d\theta$$

$$D_n^*(A, s) = \frac{1}{\pi} \int_0^{2\pi} D(A \cos \theta, -\omega A \sin \theta, s) \sin \theta \sin n\theta \, d\theta \quad (17)$$

The term $D_0^* = 0$, and C_0^* is one half of the expression obtained by setting $n = 0$ in the above equation for C_n^* . In Eq. (16) the following notation for total and partial derivatives is used:

$$(\quad)' = d/ds, \quad (\quad)_s = \partial/\partial s, \quad (\quad)_A = \partial/\partial A$$

The frequency, $\theta' = \omega$, is frequently called the static stability parameter and the amplitude decrement, $\delta = A'/A$, will be called the dynamic stability parameter. It is important to note that the ratio δ/ω must be small, $\ll 1$, for the analysis to apply. In the linearized theory for constant coefficients and for larger damping, the true frequency becomes $\omega = \sqrt{\mu(C_1 - \mu D_1^2/4)}$ and it is possible to modify the analysis of the appendix to account for this effect in the first approximation. We also note that in order for the perturbation U/A to be small, the coefficients $(C_n^*)(n^2 - 1) C_1^*$ must be small, and this assumption must be checked individually in each case.

COMPARISON OF RESULTS FOR LINEARIZED THEORY

Several authors have investigated the linearized longitudinal oscillations of reentry missiles and notable among these are Allen (1957) and Sommer and Tobak (1959). Allen's work applies to the reentry of high speed ballistic missiles and is based essentially on Eq. (3), or its dimensionless form, Eq. (14), which assumes an exponential density variation with altitude, a straight line trajectory, constant coefficients, and the drag relation $V'/V = -\mu C_D$. The application of the general formula of Eq. (16) to Eq. (14) for $C_1^* = C_1$, $D_1^* = D_1$ yields the following results:

$$\alpha \approx A \cos \theta, \quad \theta'(r) = \omega(r) = \sqrt{C_1} e^{-\epsilon K r/2} = \sqrt{-\sigma C_{m\alpha}} e^{-\epsilon K r/2} \quad (18)$$

$$\frac{A'(r)}{A(r)} = \frac{\epsilon K}{4} - \frac{\epsilon D_1}{2} e^{-\epsilon K r} \quad (19)$$

where $D_1 = C_{L\alpha} - C_D - \sigma(C_{m\alpha} + C_{m\dot{q}})$ and the other parameters are given by Eq. (15). By noting the fact that $\epsilon K r = \beta(y - y_0)$ and dividing Eq. (19) by $\omega = \theta'$, we obtain the following dimensionless amplitude equation which gives the damping per cycle and must be small for the analysis to apply:

$$\frac{1}{A} \frac{dA}{d\theta} = \frac{\delta}{\omega} \approx \frac{1}{4} \frac{\beta L \sin \gamma}{\sqrt{\mu_0 C_1}} e^{1/2 \beta y} - \frac{D_1}{2} \sqrt{\frac{\mu_0}{C_1}} e^{-1/2 \beta y} \quad (20)$$

For most reasonable vehicles the second term on the right is well behaved and small for all altitudes above sea level, whereas the first term on the right is unbounded as $y \rightarrow \infty$. What this means, of course, is that for altitudes above a certain maximum the solution method breaks down and an oscillatory motion ceases to exist. What essentially happens above this altitude is that the aerodynamic restoring moment is negligible and insufficient to prevent tumbling so that once set into motion the missile will continue to tumble end over end unless controlled by some other means. By equating the first term on the right in Eq. (20) equal to $1/4$ (a fairly small parameter), we may obtain the altitude y_r above which the motion essentially ceases to be oscillatory,

$$y_r = \frac{2}{\beta} \ln \left[\frac{\sqrt{\mu_0 C_1}}{\beta L \sin |\gamma|} \right] \quad (21)$$

We shall call this altitude the altitude of dynamic response. On the other hand, at low altitudes the right-hand term involving D_1 becomes large and may dominate the density effect term. The altitude, y_s , at which these two terms become equal in absolute value is an indication of the relative strength or importance of conventional damping in affecting the motion. This altitude will be called the altitude of dynamic stability and is given by the following:

$$y_s = \frac{1}{\beta} \ln \left[\frac{2\mu_0 |D_1|}{\beta L \sin |\gamma|} \right] \quad (22)$$

By setting $\eta = \beta(y_r - y)$, $\eta_0 = \beta(y_r - y_s)$ in the amplitude and frequency equations or Eqs. (18), (19) and integrating, we may derive the following approximate solution for α

$$\alpha \approx e^{\pm e^{(\eta-\eta_0)^4}} [A_0 e^{-\eta/4} \cos (2e^{\eta/2} + \theta_0)] \quad (23)$$

In this equation the (+) sign applies to negative values of D_1 , etc.

Now Allen (1957) has obtained an approximate solution to Eq. (14) in terms of the Bessel functions J_0 , Y_0 , which is valid at all altitudes. For the initial conditions ($\alpha = \alpha_0$, $\alpha' = 0$) at $y = \infty$, this solution can be written

$$\alpha = e^{\pm e^{\eta-\eta_0/4}} \{ \alpha_0 J_0(2e^{\eta/2}) \} \quad (24)$$

Now for an appropriate choice of A_0 and θ_0 , (i.e., $A_0 = \alpha_0 \sqrt{2/\pi}$, $\theta_0 = -\pi/4$) in Eq. (23), it is easily shown that the term in brackets in Eq. (23) corresponds to the first term in the asymptotic expansion of J_0 . Eq. (24) and the asymptotic envelope, $\sqrt{2/\pi} e^{-\eta/4}$, have been plotted in Fig. 2 for zero damping (i.e., $\eta_0 = \infty$). Also shown are the envelopes for various values of η_0 corresponding to negative damping (i.e., $D_1 < 0$). It is interesting to note in this figure how the high altitude response of a reentry vehicle is correlated by a single parameter, the altitude of dynamic response, y_r .

For vehicles transverse a curved flight path, and at low altitudes and velocities where the high-speed approximation $V'/V \approx -\mu C_D$ ceases to be valid, Allen's results do not apply. Sommer and Tobak (1959) have investigated this more general problem and, in particular, have succeeded in obtaining an approximate solution to Eq. (13) for linearized aerodynamics [or $C(\alpha, s) = C_1(s)$, $D(\alpha, \alpha', s) = D_1(s)$]. A modified W.K.B. approximation technique is used by these authors and their results can be expressed using the notation of this paper as follows:

$$\alpha = A \cos \theta, \quad \theta' = \sqrt{\mu C_1}, \quad \frac{A'}{A} = \delta \approx - \left\{ \frac{(qC_1)'}{4qC_1} + \frac{\mu}{2} [C_{L\alpha} - \sigma(C_{m_q} + C_{m_{\alpha'}})] \right\} \quad (25)$$

In this equation q is the dynamic pressure and by noting that for linearized aerodynamics the D_1 of Eq. (16) can be written $D_1 = V'/\mu V + C_{L\alpha} - \sigma(C_{m_q}$

+ $C_{m\alpha}$), [e.g., see Eq. (12)], it is easily seen that Eq. (16) can be reduced to the form of Sommer and Tobak.

In level steady flight, $(qC_1)' = 0$ and Eq. (25) reduces to the familiar result for the short period oscillations of aircraft with small damping. It is interesting to note that it is really the derivative of dynamic pressure q and not the density μ that forms the contribution of time varying parameters to the dynamic stability of vehicles.

THE EFFECT OF VARYING AIR DENSITY ON THE NONLINEAR DAMPING AND LIMIT MOTIONS OF VEHICLES

For vehicles in nearly level flight, or at low altitudes where the density variation along the flight path is negligible, the nonlinear amplitude relation in Eq. (16) becomes time-independent or autonomous and its analysis is considerably simplified. Under these conditions, Murphy (1957) has treated extensively the more general case of spinning symmetric missiles and, more specifically, has ascertained and investigated the existence of stable and unstable limit motions or periodic solutions. In the one dimensional planar case under consideration here, the limit motions become limit cycles and were investigated originally by Poincare and then Van der Pol. The question before us here is what effect, if any, do the slowly varying parameters have on these limit motions? In order to fix ideas and conclusions, we will consider a simple example consisting of a vehicle with a linear static moment, i.e., $C = C_1$, and a Van der Pol type nonlinear damping moment, i.e., $D = D_1 + D_3\alpha^2$. In addition, we will make Allen's assumptions as to the character of the trajectory (i.e., straight-line, $V'/V \approx$

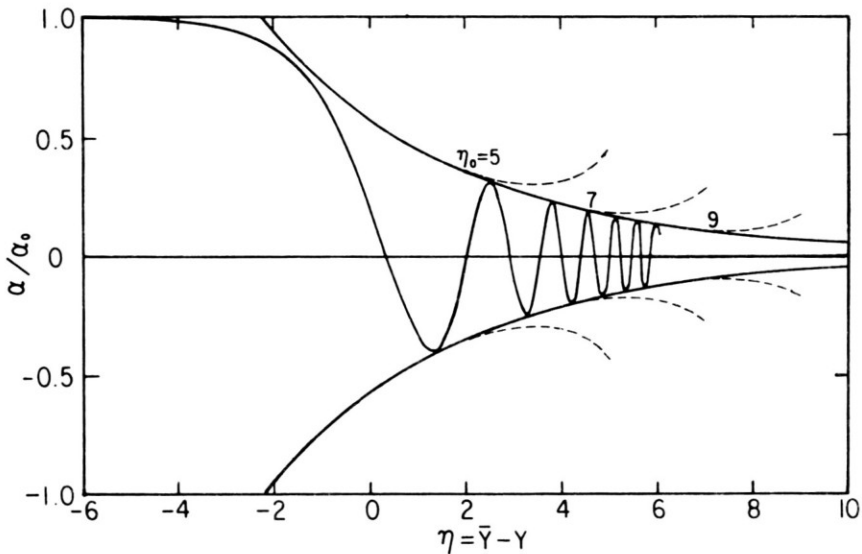


Fig. 2.

$-\mu(\dot{D}, \mu = \mu_0 e^{-\beta y})$. In this case Eq. (14) is applicable, and for simple Van der Pol damping, it reduces to the following equation:

$$\alpha''(r) + e^{-\epsilon Kr} C_1 \alpha(r) = -\epsilon e^{-\epsilon Kr} (D_1 + D_3 \alpha^2) \alpha'(r) \quad (26)$$

Application of the equations of the first approximation to Eq. (26) gives the following result for the amplitude equation:

$$\frac{A'(r)}{A(r)} = \epsilon \left\{ \frac{K}{4} - \frac{1}{2} e^{-\epsilon Kr} (D_1 + \frac{1}{4} D_3 A^2) \right\} = \delta_1 + \delta_3 A^2 \quad (27)$$

Now for $K = 0$, this equation reduces to the classic result of Van der Pol or Krylov and Bogoliubov (1943). For $D_1 < 0$, $D_3 > 0$, a stationary oscillation, or limit cycle, exists with amplitude $A^2 = A_{L_0}^2 = -\delta_1/\delta_3 = -4D_1/D_3$.

We will now investigate the case of varying air density in which the parameters δ_1 and δ_3 are no longer constant. For this purpose we note that Eq. (27) can be put into the form $A' - \delta_1 A = \delta_3 A^3$ which will be recognized immediately as a Bernoulli equation. By multiplying by $1/A^3$, Equation (27) can be rewritten as:

$$\left(\frac{1}{2\delta_1} \frac{d}{dr} + 1 \right) \left(\frac{1}{A^2} \right) = -\frac{\delta_3(\epsilon Kr)}{\delta_1(\epsilon Kr)} = \frac{1}{A_{L_0}^2(\epsilon Kr)} \quad (28)$$

and will be recognized as a linear equation for A^{-2} . The term $A_{L_0}(\epsilon Kr)$ is the stationary amplitude that would obtain if $\delta_3 < 0$, $\delta_1 > 0$ and if the dependence on ϵKr were neglected. This equation is easily integrated in closed form, and the result for A is given as follows:

$$A = A_0 e^{\phi/2} \left[1 - A_0^2 \int_0^\phi (\delta_3/\delta_1) e^\psi d\psi \right]^{-1/2} \quad (29)$$

where $\phi = \phi(r) = \int_0^r 2\delta_1(r) dr$, and where A_0 is the initial amplitude. This equation can be treated by a number of numerical methods, but perhaps it is more useful to integrate Eq. (27) directly on an analog or digital computer as Murphy (1957) does in the extensive application of his Amplitude Plane. We note by observing Eq. (28) that there are two parts to the solution, namely, homogeneous and inhomogeneous parts. In the autonomous case, the inhomogeneous solution, $A = A_{L_0}$, represents the stationary oscillation amplitude if $\delta_3/\delta_1 < 0$.

In the nonautonomous case we will denote this inhomogeneous part as the limit solution, and we note that it represents the oscillation amplitude a vehicle would have if traveling in disturbed flight for some time. This solution can be expressed formally as follows:

$$A_{L^{-2}} = -e^{-\phi} \int (\delta_3/\delta_1) e^\phi d\phi = \frac{-1}{1 + D} \frac{\delta_3}{\delta_1} \quad (30)$$

and where the indefinite integral is shown to indicate the neglect of the integration constant. D is the differential operator $d/d\phi$. By substituting the specific

expressions for δ_3 and δ_1 of Eq. (27) into Eq. (30), we may obtain the following solution for A_L^{-2} :

$$A_L^{-2} = -\frac{1}{4} \frac{D_3}{D_1} \left(1 - \frac{2x^2}{1-x} \frac{d}{dx} \right)^{-1} (1-x)^{-1} \tag{31}$$

where $x = Ke^{Kt}/2D_1 = -(sgnK)e^{\beta(y-y_s)}$ for $D_1 < 0$ and where y_s is the altitude of dynamic stability given by Eq. (22). For small values of x , the solution to Eq. (31) may be expanded in power series of x . Neglecting terms of higher order than x , we obtain the following result for the time-dependent limit cycle:

$$A_L \approx \tilde{A}_{L_0} (1 - \frac{1}{2}x) = \tilde{A}_{L_0} [1 + \frac{1}{2} (sgnK)e^{\beta(y-y_s)}] \tag{32}$$

and where A_{L_0} is the value of the stationary limit cycle amplitude for $K = 0$ (or $y_s = \infty$). It is interesting to note that this result shows that to a first approximation the time dependent limit cycle is equivalent to the case in which δ_1 and δ_3 are considered constants. It is also seen that the limit amplitude is smaller in descent and greater in ascent than the idealized value A_{L_0} . For altitudes above y_s , there is apparently no simple expression for A_L , and the complete equation must be considered.

THE EFFECTS OF NONLINEAR STATIC MOMENTS

In this part we will briefly discuss the effect of nonlinear static moments on the short period oscillations of ascending or descending vehicles. Again, in order to fix ideas and conclusions, we will assume a specific form for the static moment. The simplest form that still displays most of the important characteristics of asymmetric vehicles is the following one which contains quadratic and cubic nonlinearities in α .

$$C(\alpha,s)\alpha = C_1\alpha + C_2\alpha^2 + C_3\alpha^3 \tag{33}$$

No specific form for the damping moment $D(\alpha,\alpha',s)$ will be assumed, and thus by substituting Eq. (33) into Eqs. (16 and 17), we obtain the following first approximation equations for a vehicle with linear plus quadratic and cubic static moments and negligible damping.

$$\alpha \approx A \left\{ \cos \theta + \frac{1}{C_1 + \frac{3}{4} C_3 A^2} \left[-\frac{1}{2} C_2 A (1 - \frac{1}{3} \cos 2\theta) + \frac{1}{32} C_3 A^2 \cos 3\theta \right] \right\}$$

$$\theta' = \omega \approx \sqrt{\mu(C_1 + \frac{3}{4} C_3 A^2)}, \quad \frac{A'}{A} = \delta \approx -\left(\frac{8C_1 + 6C_3 A^2}{8C_1 + 9C_3 A^2} \right) \frac{(\mu C_1)_s}{4\mu C_1} \tag{34}$$

In discussing these equations, we would like to first point out that to a first approximation the asymmetrical components (i.e., C_2 , etc.) in the static moment

do not affect the oscillation frequency or decrement. This is due to the fact that any modification in the oscillation characteristic on one side of the origin, $\alpha = 0$, is reversed in sign on the other side by the asymmetric components, and thus the average value of these effects over a cycle is canceled out. The average value over a cycle of the symmetric nonlinearities is not zero, and this explains their appearance in the frequency and amplitude equations. We do note, however, that in the expression for α the situation is different with the asymmetric nonlinearity, C_2 , playing a somewhat more dominant role than the symmetric one, and even giving rise to a nonzero mean value to the oscillation amplitude. This mean value for α is

$$\alpha_m = -A \left[\frac{C_2 A / 2}{C_1 + \frac{3}{4} C_3 A^2} \right] \quad (35)$$

And if the oscillation amplitude is finite over a significant portion of the trajectory, then the asymmetric component may give rise to an induced lift which may cause the vehicle to depart from its nominal trajectory.

The second point we would like to make is that although large symmetric nonlinearities (i.e., $C_3 A^2$) may exist, this does not necessarily imply that the approximation method should break down. For example, if $C_1 = C_2 = 0$ and $C_3 > 0$, the approximate solutions for α and ω become

$$\alpha = A \left\{ \cos \theta + \frac{1}{24} \cos 3\theta \right\} \quad \omega = \sqrt{\frac{3}{4} \mu C_3} A \quad (36)$$

In the Appendix, the exact solution for $A = \text{constant}$, $C_2 = 0$, in terms of Jacobian elliptic functions, is discussed and compared with the approximate result. It is found that even when $C_1 = 0$ and $C_3 > 0$ (i.e., the above case) the errors in the approximate results for α and ω are less than 2 percent. This result clearly demonstrates the increased power of this method over that of Krylov and Bogoliubov since their method completely fails in this case (i.e., their perturbation term, being proportional to $C_3 A^2 / C_1$ rather than to $C_3 A^2 / (C_1 + \frac{3}{4} C_3 A^2)$ as in our case, becomes infinite as $C_1 \rightarrow 0$). In addition, we have compared in Fig. 3 the approximate results of this paper and those of Krylov and Bogoliubov for the frequency (i.e., $\omega = \sqrt{\mu(C_1 + \frac{3}{4} C_3 A^2)}$ and $\omega \approx \mu(C_1 + \frac{3}{8} C_3 A^2 / C_1)$) with the exact elliptic function solution. It is found there that the approximate result of this paper is considerably more accurate than the Krylov-Bogoliubov approximation for all values of $C_3 A^2 / \mu C_1 = \epsilon A^2$.

Our final comment on the effects of nonlinear static moments applies to the amplitude equation. Here we find that the effect is to multiply the linear decrement, that is, the value the decrement δ would have if no static nonlinearities existed, by a term nonlinear in A^2 which reduces to one for small amplitudes or small perturbations (i.e., $C_3 A^2 / C_1$). This result also differs somewhat from that of Krylov and Bogoliubov whose result does not contain this multiplicative factor. Murphy (1962) has arrived at the same result as we have through the application of his "quasi linear substitution" method and has compared this result with the results of his "perturbation" method and those of the exact

numerical integration of the equation of motion. He finds that both analytical results predict the correct qualitative behavior for this multiplicative factor but that his perturbation method is considerably more accurate. However, we must state that we do not see how this superior method is capable of being extended (1) to more general nonlinearities in which an exact solution to the equation of conservative motion is not known, or (2) to predicting the effect of static moment perturbations in the nonlinear frequency relation. In addition, we might add that the result of Eq. (34) shows much better correlation with $8C_1 + 9C_3A^2$ replaced by $8(C_1 + C_3A^2)$. There seems to be no theoretical justification for making this simple change, but we believe that it should be possible to get a more accurate correlation by resorting to the higher approximations.

EFFECT OF LIFT ON THE PHUGOID OSCILLATIONS AT HIGH SPEEDS

We will now investigate the long period oscillations of a high-speed missile that is operating at a nearly constant lift coefficient with a propulsive thrust that nearly balances the drag force. This corresponds to the classical low speed phugoid oscillations that are associated with negligible angular pitching moments, because of the slowness of the oscillations, and a negligible resultant drag force. This classical long period oscillation corresponds to a direct exchange between the kinetic energy and the potential energy at the constant lift coefficient corresponding to the trimmed steady state value for zero pitching moment.

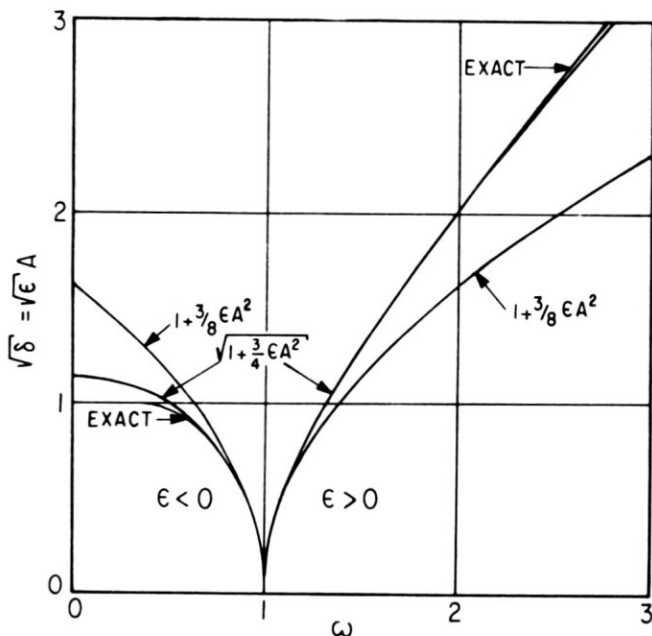


Fig. 3.

Since we are assuming a nearly constant lift coefficient, it is to our advantage to use the axes system that is always tangent to the flight path, as shown in Fig. 1, so that if we include the variation in the gravitational attraction we can write Eq. (1) for this application as

$$\begin{aligned}
 m\dot{V} &= T - \frac{1}{2} \rho V^2 SC_D - mg \left(\frac{R}{r} \right)^2 \sin \gamma \approx -mg \left(\frac{R}{r} \right)^2 \gamma \\
 mV\dot{\gamma} &= \frac{1}{2} \rho V^2 SC_L - m \left[g \left(\frac{R}{r} \right)^2 - \frac{V^2}{r} \right] \cos \gamma \\
 &\approx \frac{1}{2} \rho V^2 SC_L - m \left[g \left(\frac{R}{r} \right)^2 - \frac{V^2}{r} \right] \\
 B(\ddot{\gamma} + \ddot{\alpha} - \ddot{\phi}) &= \frac{1}{2} \rho V^2 SLC_m \approx 0
 \end{aligned} \tag{37}$$

Then we can linearize Eq. (37) for the long period oscillations in the following manner:

$$\begin{aligned}
 \frac{\dot{r}}{V} &= \sin \gamma \approx \gamma; \quad \left(\frac{\ddot{r}}{V} - \frac{\dot{r}\dot{V}}{V^2} \right) = \dot{\gamma} \cos \gamma \approx \dot{\gamma} \\
 r(t) &= R[1 + \epsilon(t)]; \quad V(t) = U[1 + \delta(t)] \\
 \rho(r) &= \rho(R) \left[1 + \frac{\rho'}{\rho} \epsilon R \right]; \quad \rho' = \frac{\partial \rho}{\partial r} \\
 U\dot{\delta} &\approx -\frac{gR}{U}\epsilon; \quad \delta \approx -\frac{gR}{U^2}\epsilon + \text{constant} \\
 \dot{\epsilon} &\approx \frac{g}{R} \left[\frac{L_0}{mg} \left(\frac{\rho'}{\rho} R\epsilon + 2\delta \right) + 2\epsilon + \frac{U^2}{gR} (2\delta - \epsilon) \right] \\
 &\approx -\frac{g}{R} \left[\frac{L_0}{mg} \left(\frac{2}{F^2} - R\frac{\rho'}{\rho} \right) + F^2 \right] \epsilon + \text{constant} \\
 \frac{L_0}{mg} &= \frac{\rho U^2 SC_L}{2mg} = \left(1 - \frac{U^2}{gR} \right) = (1 - F^2); \quad F^2 = \frac{U^2}{gR}
 \end{aligned} \tag{38}$$

Therefore, the long-period or phugoid oscillation with a constant lift coefficient has a period of

$$\begin{aligned}
 T &= 2\pi \left\{ \frac{g}{R} \left[F^2 + \frac{L_0}{mg} \left(\frac{2}{F^2} - R\frac{\rho'}{\rho} \right) \right] \right\}^{-1} \\
 &= 2\pi \frac{R}{U} \left\{ 1 + \frac{L_0}{mg} \frac{1}{F^2} \left(\frac{2}{F^2} - R\frac{\rho'}{\rho} \right) \right\}^{-1} \\
 &= \sqrt{2} \pi \frac{U}{g} \left\{ 1 + \frac{U^2}{2g} \left[-\frac{\rho'}{\rho} (1 - F^2) - \frac{1}{R} (2 - F^2) \right] \right\}^{-1}
 \end{aligned}$$

$$F^2 = \frac{U^2}{gR} = \text{constant} \approx \left(\frac{\text{ft/sec}}{26,400} \right)^2 < 1 \quad (39)$$

The last expression for the period shows that as the speed decreases we eventually attain the classical low speed phugoid value of $\sqrt{2} \pi (U/g)$, but for all supersonic speeds this classical value is greatly decreased by the usual atmospheric density gradient. For example, in the earth's atmosphere

$$\frac{\rho'}{\rho} \approx - \frac{1}{22 \times 10^3} \text{ft}^{-1} > > \frac{1}{R} \approx \frac{1}{21 \times 10^6} \text{ft}^{-1} \quad (40)$$

Consequently, at an average flight speed of 1,500 ft/sec any long period oscillation is decreased by a factor of 0.623 from the classical low-speed value of $\sqrt{2} \pi (U/g)$, while at a speed of 10,000 ft/sec it is decreased by a factor of 0.12. This effect is nearly all produced by the atmospheric density gradient. It is only after the flight speeds have nearly attained orbital speed, i.e., $F \rightarrow 1$, that effect of the change in the gravitational force has any significance.

The simple expression for the phugoid period that is given in Eq. (39) is in good agreement with the numerical values calculated by Etkin (1961) for either the constant thrust rocket engine or the air-breathing engine whose thrust varied with the density. Equation (39) is also valid at or near orbital speeds, and it proves that a resultant lift force always decreases the orbital period of a satellite since $\rho'/\rho < 0$.

If we include a resultant drag force then we can also obtain an aerodynamic damping which is given by

$$\exp \left\{ - \frac{L_0}{mU} \frac{C_D}{C_L} - \frac{U \left(- \frac{\rho'}{\rho} \right) \frac{C_D}{C_L}}{R \left(- \frac{\rho'}{\rho} \right) + \frac{2}{F^2} + \left(\frac{2m}{\rho SRC_L} \right)} \right\} t$$

$$L_0 = mg (1 - F^2); \quad F^2 = \frac{U^2}{gR} \leq 1 \quad (41)$$

APPENDIX

ANALYSIS FOR THE SHORT-PERIOD OSCILLATIONS

We will now outline the mathematical technique which is used to analyze Eq. (13) for the nonlinear time-dependent oscillations of unpowered vehicles on arbitrary paths through the atmosphere.

The type of system we will analyze has the following form:

$$\alpha''(s) + \omega_0^2(s)\alpha = f(\alpha, \alpha', s) \quad (42)$$

where f and ω_0 are slowly varying functions of s , that is, $\partial\omega_0/\partial s$ and $\partial f/\partial s$ are small quantities with their higher order derivatives being small quantities of

order $\partial^n f / \partial s^n = 0$ ($\partial f / \partial s$)ⁿ. It is also assumed that the function f can be split into conservative and nonconservative parts, i.e., $f(\alpha, \alpha', s) = g(\alpha, s)\alpha + h(\alpha, \alpha', s)\alpha'$, and that a periodic solution to Eq. (42) exists for $h \equiv 0$, for $s = s_0 = \text{constant}$ in $\omega_0(s)$ and $g(\alpha, s)$ and for some domain of initial conditions $(\alpha, \alpha') = (\alpha_0, \alpha'_0)$ encircling but not necessarily including the origin ($\alpha = 0, \alpha' = 0$). Finally, it is assumed that the nonconservative part of $f(\alpha, \alpha', s)$, i.e., $h(\alpha, \alpha', s)$, is small with respect to the conservative part of the equation or $\omega_0^2(s)\alpha - g(\alpha, s)\alpha$.

To facilitate the analysis, we shall rewrite Eq. (42) in first-order form by taking $\alpha = \alpha_1, \alpha_2 = \alpha' = \alpha'_1$. We will then have the two first-order equations:

$$\alpha'_1 = \alpha_2, \quad \alpha'_2 = -\omega_0^2(s)\alpha_1(s) + f(\alpha_1, \alpha_2, s) \quad (43)$$

The essential theme of the analysis of this equation is to introduce a transformation of variables $(\alpha_1, \alpha_2) \rightarrow (A, \theta)$ which is dependent upon several arbitrary functions (ω_1, U, V) . The system of differential equations for α_1 and α_2 thus transform into a system for A and θ . The problem then becomes one of choosing ω_1, U , and V in such a manner that these transformed equations are simplified and are thus made easier to analyze. The details of this procedure will become clear in the following.

The specific form of the transformation is

$$\alpha_1 = A \cos \theta + U(A, \theta, s), \quad \alpha_2 = -\omega_1(A, s) [A \sin \theta + V(A, \theta, s)] \quad (44)$$

where U and V are periodic in θ with period $T = 2\pi$. By setting $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, $\sin \theta = 1/2i(e^{i\theta} - e^{-i\theta})$ and $U + iV = Z(A, \theta, s) e^{i\theta}$, substituting these and Eq. (44) into Eq. (43), adding them, and then after some algebraic manipulation, we may obtain the following complex relation between A', θ' , and ω_1, Z and f :

$$\begin{aligned} & 2\omega_1\{Z\theta' + (1 + Z_A)A' + Z_s\} + \{[\omega_1' + i(2\omega_1\theta' - \omega_1^2 - \omega_0^2)](A + Z)\} \\ & = \{[\omega_1' - i(\omega_1^2 - \omega_0^2)](A + \bar{Z})e^{-2i\theta} - zie^{-i\theta}f(A + Z, A + \bar{Z}, \theta, s)\} \quad (45) \end{aligned}$$

In this equation $()' = d/ds$, $()_s = \partial/\partial s$, $()_A = \partial/\partial A$, and $()_\theta = \partial/\partial \theta$.

Now on account of the arbitrary nature of ω_1 and Z , we have several alternatives open for the analysis of Eq. (45). One alternative is the classic variation of parameters method whereby Z is chosen identically zero, and ω_1 taken equal to ω_0 . Eq. (45) is considerably simplified, and by separating real and imaginary parts, two separate equations for A' and θ' can be obtained which have the form

$$A' = \delta(A, \theta, s)A, \quad \theta' = \omega(A, \theta, s) = \omega_\theta(s) + \Omega(A, \theta, s) \quad (46)$$

It will be noticed that these equations contain A and θ and, thus, being unseparated in these variables, represent no essential simplification with regard to integrability. Although several procedures are available for the analysis of Eq. (46), it has been found preferable to work directly with Eq. (45) and, more specifically, to attempt to assign or determine $Z(A, \theta, s)$ in such a way that the

transformed differential equations, Eq. (46), are independent of θ . We, therefore, set $A' = \delta(A, s)A$ and $\theta' = \omega(A, s)$ in Eq. (45) and after rearranging obtain the following equation for $Z(A, \theta, s, \omega_1)$:

$$Z_\theta = -\frac{1}{2\omega\omega_1} \{2\omega_1[(1 + Z_A)\delta A + Z_s] + [\omega_1' + i(2\omega_1\omega - \omega_1^2 - \omega_0^2)](A + Z) - [\omega_1' - i(\omega_1^2 - \omega_0^2)](A + \bar{Z})e^{-2i\theta} + 2ie^{-i\theta}f(A + Z, A + \bar{Z}, \theta, s, \omega_1)\} \quad (47)$$

Now for the analysis of this equation there are again several alternatives available. However, most of them depend upon the following order of magnitude restrictions which enable Z to be put equal to zero as a first approximation in the right-hand side of Eq. (47) and which we will assume to be satisfied in further calculations:

$$(|Z|/A, \quad |Z_A|, \quad \omega_1|Z_s|/|f(A, \theta, s)|) \ll 1 \quad (48)$$

In this equation, $f(A, \theta, s)$ is $f(A + Z, A + Z, \theta, s)$ with Z set equal to zero. Thus, taking $Z = 0$ in Eq. (47), we obtain the following equation for the first approximation to Z_θ :

$$Z_\theta = -\frac{1}{2\omega\omega_1} \{[(2\omega_1\delta + \omega_1') + i(2\omega_1\omega - \omega_1^2 - \omega_0^2)]A - [\omega_1' - i(\omega_1^2 - \omega_0^2)]Ae^{-2i\theta} + 2ie^{-i\theta}f(A, \theta, s)\} \quad (49)$$

Now in this equation we have the two undetermined functions δ and ω which are the expressions for A'/A and θ' and also the undetermined parameter ω_1 . By expanding f into complex Fourier series, $f = \sum F_n e^{in\theta}$, $F_n = 1/2\pi \int_0^{2\pi} f(\) e^{-in\theta} d\theta$, and imposing the condition that Z be periodic in θ (which is essentially the condition that the mean value over a period of Eq. (49) be zero), we may integrate Eq. (49) w.r.t. θ and obtain the following relations for the first approximations to δ , ω , and Z .

$$\begin{aligned} \theta' = \omega &\approx \frac{A(\omega_1^2 + \omega_0^2) - 2F_{1R}(A, s, \omega_1)}{2\omega_1 A}, \quad \frac{A'}{A} = \delta \\ &\approx -\frac{\omega_1' A - 2F_{1I}(A, s, \omega_1)}{2\omega_1 A} \end{aligned} \quad (50)$$

$$Z = (U + iV)e^{-i\theta} \approx Z_0(A, s) + \frac{1}{2\omega\omega_1} \left\{ \frac{1}{2} [i\omega_1' + (\omega_1^2 - \omega_0^2)] A e^{-2i\theta} - \sum_{-\infty}^{\infty} \frac{2}{n-1} F_n e^{i(n-1)\theta} \right\} \quad (51)$$

F_{nR}, F_{nI} are the real and imaginary parts of F_n respectively, and we note also that F_n is a function of ω_1 . In Eq. (51) $Z_0(A, s)$ is an arbitrary function of (A, s) which essentially is an integration constant of Eq. (49). The \sum in Eq. (51) omits the term $n = 1$.

Now in order that Eqs. (50) and (51) be valid approximations, the assumptions of Eq. (48) must be satisfied. The precise calculation and comparison of terms requires considerable analysis and will not be gone into here. If these conditions are satisfied, then higher approximations may be obtained by perturbation or iteration of Eq. (47).

We will return to the real variables α and α' by substituting Eq. (51) into Eq. (44). Since Z_0 is arbitrary, we shall choose it such that the coefficients of $\cos \theta$ and $\sin \theta$ in the perturbation or summation term of α vanish, and we are thus left with

$$\alpha = A \cos \theta + \frac{1}{\omega \omega_1} \left\{ F_0 + \sum_{n=2}^{\infty} \frac{2}{1-n^2} (F_{nR}(A,S) \cos n\theta - F_{nI}(A,S) \sin n\theta) \right\} \quad (52)$$

A similar but more complicated result for α' is obtained but will not be written here. In Eq. (52) the coefficients F_{nR} , F_{nI} are related to the coefficients in the real expansion of f in terms of $\sin n\theta$ $\cos n\theta$ as follows:

$$f = A_0 + \sum_1^{\infty} (A_n \cos n\theta + B_n \sin n\theta) = \sum_{-\infty}^{\infty} (F_{nR} + iF_{nI}) e^{in\theta}$$

$$F_0 = A_0, \quad F_{nR} = F_{-nR} = \frac{1}{2}A_n, \quad F_{nI} = -F_{-nI} = \frac{1}{2}B_n$$

Now, up until this point we have left the choice of ω_1 completely arbitrary. We will now consider two important choices for ω_1 .

$$(1) \quad \omega_1 = \omega_0 \quad \delta \approx -\frac{\omega_0'}{2\omega_0} + \frac{1}{2\omega_0 A} B_1(A, s, \omega_0), \quad \omega \approx \omega_0(s) - \frac{1}{2A\omega_0} A_1(A, s, \omega_0) \quad (53)$$

$$(2) \quad \omega_1 = \omega \quad \delta = -\frac{\omega'}{2\omega} + \frac{1}{2\omega A} B_1(A, s, \omega), \quad \omega^2 \approx \omega_0^2(s) - \frac{1}{A} A_1(A, s, \omega) \quad (54)$$

In case (1) we have the results of Krylov and Bogoliubov (1943) for $\partial\omega_0/\partial s = \partial f/\partial s = 0$ and that of Bogoliubov and Mitropolsky (1955) for $\partial\omega_0/\partial s$ and $\partial f/\partial s$ small quantities. For the approximation to be valid in this case, $B_1/2\omega_0^2 A$, $A_1/2\omega_0^2 A$ must be small with respect to 1. Thus, it is seen that this Krylov-Bogoliubov-Mitropolsky approximation breaks down as the unperturbed frequency ω_0 becomes small of the order of A_1/A or B_1/A . Now it is known that although the unperturbed frequency ω_0 may become small or vanish in Eq. (42), there is a class of functions F and initial conditions α_0 , α_0' , such that a periodic solution about the origin exists. This is the conservative dynamical system where f has the form $f = g(\alpha)\alpha$ and ω_0^2 is a constant. In this case, the Fourier expansion of the exact solution can be written as a series in $\cos n\theta$ where $\theta' = \nu(E)$, and E is the initial condition, amplitude, energy or some other constant of the motion. Now for a wide range of values of E , this expansion of the solution

is rapidly convergent and may be well approximated by the first harmonic $A(E) \cos \theta$, while on the other hand the frequency $\nu(E)$ is not at all approximated by ω_0^2 which in some cases may even be negative. In this case of the conservative dynamical system and for small nonconservative perturbations of it, an appropriate choice of ω_1 in Eqs. (50) and (51) may give a superior and more uniformly valid approximation for ω and δ . If the exact frequency, $\nu(E)$, of the conservative system is known and if E can be related to the coefficient of the first harmonic, i.e., $E = E(A)$, then an appropriate choice could be $\omega_1(A) = \nu[E(A)]$.

An alternative choice for ω_1 is that of case (2) above or Eq. (54). In this case, the specific form of ω (and ω_1) as a function of A and s is not always given explicitly since the coefficient A_1 also contains ω . In many cases, A_1 does not contain ω and in this case we have an explicit relation for ω^2 . This choice for ω_1 has been previously assumed above.

In order to illustrate the method in the case of a large conservative non-linearity and to indicate its accuracy, we will now investigate the following dynamical system:

$$\alpha''(s) + \mu(C_1\alpha + C_3\alpha^3) = 0 \tag{55}$$

where μ , C_1 , and C_3 are constants. This equation has exact solutions periodic about the origin and which can be written in terms of the Jacobian elliptic functions sn and cn as follows:

$$(1) \quad C_3 \leq 0 \quad \alpha = \alpha_0 sn(\lambda s + K(m) m)$$

$$0 \leq m = \frac{-C_3\alpha_0^2}{2C_1 + C_3\alpha_0^2} \leq 1, \quad \lambda^2 = \mu(C_1 + \frac{1}{2} C_3\alpha_0^2) \tag{56}$$

$$(2) \quad C_3 \geq 0 \quad \alpha = \alpha_0 cn(\lambda s/m)$$

$$0 \leq m = \frac{1}{2} \left(\frac{C_3\alpha_0^2}{C_1 + C_3\alpha_0^2} \right) \leq 1, \quad \lambda^2 = \mu(C_1 + C_3\alpha_0^2) \tag{57}$$

In both cases the natural frequency of oscillation is given by $\omega = [2K(m)/\pi\lambda]^{-1}$, where $K(m)$ is the complete elliptic integral of the first kind [see Milne Thomsen (1950)]. The approximate solutions for $\omega_1 = \omega_0$ and $\omega_1 = \omega$ are easily calculated and yield:

$$(1) \quad \omega_1 = \omega_0 \quad \alpha = A \left\{ \cos \theta + \frac{1}{32} \frac{C_3 A^2}{C_1} \cos 3\theta \right\}$$

$$A = \text{constant}, \theta' = \omega = \sqrt{\mu C_1} \left(1 + \frac{3}{8} \frac{C_3 A^2}{C_1} \right) \tag{58}$$

$$(2) \quad \omega_1 = \omega \quad \alpha = A \left\{ \cos \theta + \frac{1}{32} \left(\frac{C_3 A^2}{C_1 + \frac{3}{4} C_3 A^2} \right) \cos 3\theta \right\}$$

$$A = \text{constant}, \theta' = \omega = [\mu (C_1 + \frac{3}{4} C_3 A^2)]^{\frac{1}{2}} \tag{59}$$

The exact and the two approximate frequency relations of Eqs. (56)–(59) for $\mu C_1 = 1$ and $C_3 = \epsilon$ under the assumption that A equals the exact amplitude, α_0 , are compared in Fig. (3). It is seen that for $\epsilon > 0$ the Krylov-Bogoliubov approximation, Eq. (58), rapidly departs from the exact solution for values of $\sqrt{|\epsilon|} A$ greater than about $\frac{3}{4}$. On the other hand, the improved approximation of Eq. (59) gives a result which is very accurate (within 2.24 percent) of the exact result for all $\epsilon > 0$. For negative values of ϵ , the improved approximation also gives a more accurate result than the Krylov-Bogoliubov method but apparently breaks down as $\sqrt{|\epsilon|} A$ approaches one and ω becomes less than $\frac{1}{2}$. However, much of this inaccuracy is due to the fact that the parameter A was taken identically equal to the exact amplitude α_0 . If we use Eq. (59) to calculate α_0 as a function of A , and then invert this relation to obtain A as a function of α_0 , we may plot the improved approximate frequency as a function of $\sqrt{|\epsilon|} \alpha_0$.

Using the approximate inverse relation of $A \approx \alpha_0 \left(1 + \frac{1}{32} \frac{\epsilon \alpha_0^2}{1 + \frac{3}{4} \epsilon \alpha_0^2} \right)^{-1}$, we find that for $-\epsilon \alpha_0^2 = 1$, the approximate frequency is $1/7$ whereas the exact frequency vanishes. Assuming $\alpha_0 = A$, the approximate frequency at $-\epsilon \alpha_0^2 = -\epsilon A^2 = 1$ is $\frac{1}{2}$, and we thus see the increased accuracy of using the actual amplitude of the oscillation rather than the approximation $\alpha_0 = A$ in the frequency relation of Eq. (59).

We may also compare the approximate solutions for α against the exact result. It is found that although the Krylov-Bogoliubov approximation breaks down for $C_1 \rightarrow 0$, $C_3 > 0$, the improved result is regular being $\alpha = A(\cos \theta + \frac{1}{24} \cos 3\theta)$ for $C_1 = 0$. In this case, $m = \frac{1}{2}$ in the exact solution, Eq. (57), and it is found that the improved approximation for α is in error by at most 1.09 percent (at $\theta = \pi/4$ where $\alpha \text{ approx} = 0.650 \alpha_0$ and $\alpha \text{ exact} = 0.6434 \alpha_0$).

As a final note, we would like to comment on the extension of the methods described above to the analysis of nonlinear systems with more than one degree of freedom. We have performed this analysis for the equations of a spinning symmetric missile, and the results appear in a form quite similar to the one dimensional results of this section except that now there are two systems $A_1' = \delta(A_1, A_2, s) A_i$ and $\theta_i' = \omega_i(A_1, A_2, s)$, where $i = 1, 2$, in place of one. These equations are nearly identical with the results of Murphy (1957 and 1962) with the exception that the higher harmonics in the perturbation are obtained directly by our method. We have also looked into the application of these methods to the motions of asymmetric missiles. In principle the method is fairly direct but the algebra becomes very complicated, and we, as yet, have been unable to obtain results useful to the ballistician or aerodynamicist.

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DISCUSSION

Author(s): E. V. Laitone and T. J. Coakley

Discussor: E. T. Benedikt, Northrod Space Laboratories

Whenever the coefficients of the (linearized) equations of motion can be approximated by linear functions of the independent variable, the integrals are expressible in terms of Bessel functions. Indeed, some of the graphical representations of the numerically obtained solutions do suggest the behavior of such functions. Comments as to the range of validity of the above approximation would be of interest.

Author's reply to discussion:

The question is simply that of asking under what conditions can the variable coefficients in the equation of motion be replaced by their linear expansions. By referring to Eq. (3) of the paper, and assuming D_1 and C_1 constant, we find that the variation in the coefficients of the equation of motion is simply due to the variation in $\mu(s)$. Thus, if $\mu_2(s) = \mu_0 + \mu_1 s$ is the linear expansion of μ , then the relative error between this and the exact value is $\Delta = (\mu - \mu_2)/\mu$ and this provides a means for determining the accuracy of the linear expansion in the approximate equation of motion. If we assume an exponential variation in density with altitude, i.e., $\mu = \mu_0 e^{-\beta y}$, and are interested in the accuracy of the linear expansion in the vicinity of a given reference altitude, y_0 , then the relative error as written above becomes

$$\Delta(y - y_0) = 1 - [1 - \beta(y - y_0)] e^{\beta(y - y_0)}$$

This error attains a value of 0.104 or 10.4 percent for $\beta(y - y_0) = 0.4$, and 0.061 or 6.1 percent for $\beta(y - y_0) = -0.4$.

Thus, in order to keep the relative error less than about 10 percent, the altitude band over which the approximation is to apply should not exceed $2(y - y_0)_{\max} = \Delta y = 2 \times 0.4 \times 23,000$ or roughly 20,000 ft.